Noise-induced transition from translational to rotational motion of swarms

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We consider a model of active Brownian agents interacting via a harmonic attractive potential in a two-dimensional system in the presence of noise. By numerical simulations, we show that this model possesses a noise-induced transition characterized by the breakdown of translational motion and the onset of swarm rotation as the noise intensity is increased. Statistical properties of swarm dynamics in the weak noise limit are further analytically investigated.

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I. INTRODUCTION

In different natural and social systems, agents form groups characterized by cooperative coherent motion [1]. Such collective swarm motions have been observed and investigated in bacterial populations [2–6], in slime molds [7–10], for ants [11,12] and fish [13–15], in the motion of pedestrians [16], and for the car traffic [17]. To describe these phenomena, various models of collective coherent motion in populations of self-propelled or self-driven particles have been proposed (see [18,19]). Some of them are formulated in terms of interacting automata located on a lattice or having continuous coordinates [20–23]. A different class of models is based on dynamical equations for individual self-propelled particles, including velocity-dependent negative friction terms to account for their active motion and assuming that interactions between the particles are described by some binary potentials. This latter approach has been used to phenomenologically characterize motion of biological organisms [24,25], individual blood cells [26,27], and humans [16,28], and to describe the behavior of physical particles with energy depots [29–32]. Effective dynamical equations with velocity-dependent friction could be approximately derived for floating particles which propel themselves by releasing surfactant into the medium [33]. Continuum approximations to discrete particle models, hydrodynamical models of active fluids, have been proposed [19,34–38].

The common property of all swarm models is that they show the emergence of coherent collective flows starting from a disordered state with random velocity directions of individual particles (agents). Such kinetic transitions have been extensively investigated for automata systems [20,39] and in the framework of hydrodynamics [34,37]. The ordered states of swarms can represent simple translational motion or be characterized by vortex flows. Both spatially distributed and localized swarm states are possible. In a distributed state, the population fills the entire available medium. In contrast to this, the population forms compact spatial groups in a localized state. An interesting example of a localized swarm state is a rotating flock of finite extent, seen in the simulations of a discrete model of self-propelling particles and described analytically within a continuum active fluid approximation [23] (see also [19]).

Localized states of swarms may undergo transitions leading to new dynamical regimes, when the system parameters or the noise intensity are gradually changed. In a previous publication [40], a noise-induced transition from the localized state with translational motion to a state with incoherent oscillations without translational motion was investigated numerically and analytically for a one-dimensional system of interacting self-propelled particles. In the present article, we extend investigations of this system to two spatial dimensions. We study here a population of self-propelled particles interacting via a parabolic interaction potential corresponding to linear attracting forces between the pairs. In the absence of noise, this dynamical system has two kinds of attractors, corresponding, respectively, to a compact traveling state of the entire population and to a state where it rotates as a vortex without any global translational motion. The aim of our study is to investigate the effects of noise on translational swarm motion. We find that the system is highly sensitive to stochastic forces. When noise is present, the traveling swarm is a cloud of particles characterized by different dispersions in the directions parallel and transverse to the direction of translational motion. Our numerical simulations confirmed by an approximate analytical study show that the mean-square transverse dispersion of a swarm is proportional to the square root of the noise intensity, whereas its dispersion along the direction of motion depends linearly on the noise intensity. Therefore, for weak noises the swarm looks like a pancake oriented orthogonally to the motion direction. When the noise is increased, the swarm gradually acquires a more symmetric shape. For strong noise, we find that the translational motion of a swarm becomes suddenly impossible and is abruptly replaced by a rotational regime with a vortex flow. The detailed formulation of the model is presented in the next section. In Sec. III we describe the results of numerical simulations. The statistical properties of a traveling swarm in the weak noise limit are approximately explained by an analytical theory which is constructed in Sec. IV. The

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paper ends with conclusions and discussion of the obtained results.

II. THE MODEL

We consider a swarm formed by N identical self-propelled particles of unit mass interacting via an attractive parabolic pair potential. The dynamics of the system is given by the following set of evolution equations:

$$\dot{\mathbf{r}}_i = \mathbf{v}_i, \tag{1a}$$

$$\dot{\mathbf{v}}_i = \mathbf{F}_i - \frac{a}{N}\sum_{j=1}^{N}(\mathbf{r}_i - \mathbf{r}_j) + \xi_i(t). \tag{1b}$$

The forces $\mathbf{F}_i$ depend on the particle velocity and are introduced to take into account active motion of particles. We choose them in the form

$$\mathbf{F}_i = (1 - \mathbf{v}_i^2)\mathbf{v}_i, \tag{2}$$

so that, in the absence of noise and interactions, the particle acquires the unit velocity $v=1$. Additionally, the particles are subject to stochastic white forces ξ_i of strength D which are independent for different particles and are characterized by the correlation functions

$$\langle \xi_i(t)\rangle = 0, \quad \langle \xi_i(t)\xi_j(t')\rangle = 2D\delta(t-t')\delta_{ij}.$$

This model has previously been introduced in [40]. It is phenomenological, but rather general because it can be viewed as a normal form for a population of particles near a supercritical bifurcation corresponding to spontaneous onset of active motion (see [19]). In this model, attractive interactions between particles have an infinite range and grow linearly with the separation between them. Because we shall consider only spatially localized states of the population, our results will hold, however, also for the situations when interactions are characterized by a finite range, but it is much larger than the mean swarm diameter. It should be noted that, in a different context, the model (1) has been considered already by Rayleigh [41].

The study of the one-dimensional version of the model (1) has shown that, as the noise intensity D is increased, spontaneous breakdown of translational swarm motion takes place here [40]. Some statistical properties of translational swarm motion in the two-dimensional model (1) (with a slightly different choice of the forces $\mathbf{F}_i$) have subsequently been discussed [42]. For the case of two interacting particles ($N=2$), the rotational modes were described in [32], where simulations for small rotating clusters consisting of 20 particles have also been reported. The aim of the present work is to perform systematic numerical and analytical investigations of the behavior described by this two-dimensional model.

III. NUMERICAL SIMULATIONS

Numerical integration of Eqs. (1) was performed using the Euler scheme with the constant time step of 0.001. In all simulations, the total number of particles was fixed to $N=300$ and the coefficient a specifying the strength of interactions between the particles was set to $a=100$. To produce a traveling localized state of the swarm, special initial conditions were used. At time $t=0$, all particles had identical positions and velocities and the noise was switched only a little later, at time $t=30$.

Several statistical characteristics of the swarm were monitored during simulations. The center of mass $\mathbf{R}$ of the swarm and its mean velocity $\mathbf{V}$ at time t were defined as $\mathbf{R}(t)=(1/N)\Sigma_i\mathbf{r}_i(t)$ and $\mathbf{V}(t)=(1/N)\Sigma_i\mathbf{v}_i(t)$, respectively. Because the cloud of traveling particles in the presence of noise was anisotropic, we also determined its instantaneous mean-square dispersions in the directions parallel ($S_\parallel$) and orthogonal ($S_\perp$) to the direction of its instantaneous mean velocity $\mathbf{V}$. They were defined as

$$S_\parallel(t) = \frac{1}{NV^2(t)}\sum_{i=1}^{N}\{[\mathbf{r}_i(t) - \mathbf{R}(t)]\cdot\mathbf{V}(t)\}^2, \tag{3a}$$

$$S_\perp(t) = \frac{1}{NV^2(t)}\sum_{i=1}^{N}\{[\mathbf{r}_i(t) - \mathbf{R}(t)]\times\mathbf{V}(t)\}^2. \tag{3b}$$

Additionally, angular momenta of all particles with respect to the mass center were determined as

$$\mathbf{L}_i(t) = [\mathbf{r}_i(t) - \mathbf{R}(t)]\times[\mathbf{v}_i(t) - \mathbf{V}(t)]. \tag{4}$$

Figure 1(a) shows the time dependence of the magnitude $V=|\mathbf{V}|$ of the mean swarm velocity for two simulations with different noise. When the noise is relatively weak ($D=0.067$), its introduction leads to some fluctuations in the instantaneous swarm velocity and a decrease of its average level. If a stronger noise ($D=0.07$) is applied, the swarm velocity first behaves as for the weaker noise, but then abruptly drops down to a value close to zero. This sudden transition corresponds to the breakdown of translational motion of the swarm. In Fig. 1(b) we have plotted the time-averaged swarm velocity $\langle V\rangle$ as a function of the noise intensity D. The average velocity gradually decreases with noise, until the breakdown occurs at $0.067<D<0.070$.

In the state with translational motion at relatively weak noise, the direction of the swarm motion does not remain constant with time. The swarm travels along a complex trajectory, part of which is shown in Fig. 2 (such trajectories should correspond to the Brownian motion of the entire swarm). In the inset in this figure, we display the distribution of particles in the swarm at some time moment. It can be noticed that the cloud of particles is significantly squeezed along the direction of swarm motion.

Figure 3 shows the computed average longitudinal ($S_\parallel$) and transverse ($S_\perp$) dispersions as functions of the noise intensity D. For weak noise, $S_\perp \gg S_\parallel$ so that the swarm is strongly squeezed. As noise increases, the shape of the swarm becomes more symmetric and the transversal dispersion approaches the dispersion along the direction of translational motion. Finally, after the breakdown of translational motion has taken place for a sufficiently strong noise, the swarm becomes statistically circular ($S_\perp = S_\parallel$).

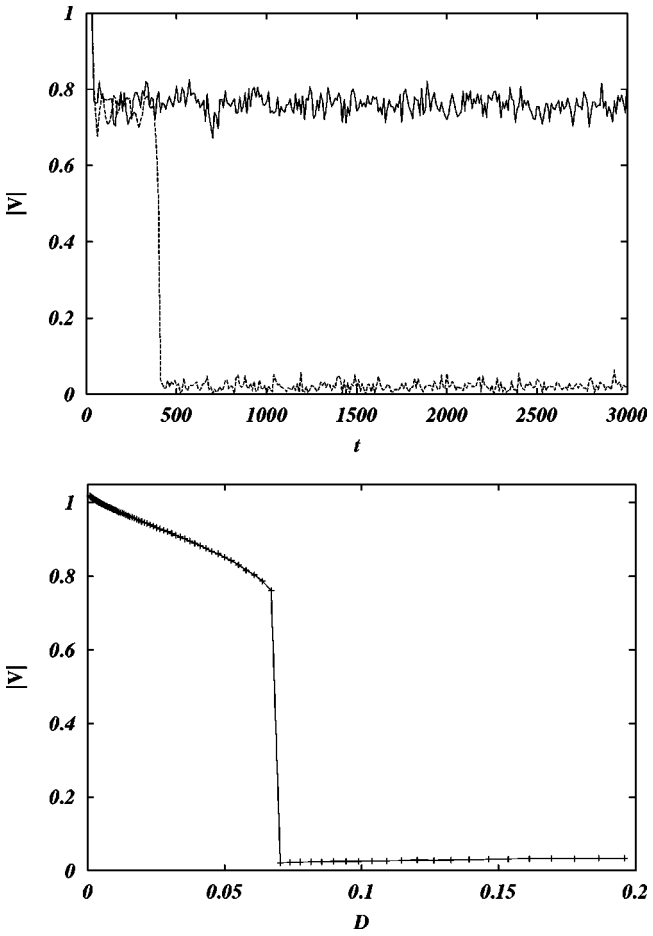

FIG. 1. (a) Time evolution of the mean velocity of a swarm before (solid line, $D=0.067$) and after (dashed line, $D=0.070$) the stochastic breakdown of the translational mode. (b) Mean velocity of the swarm with increasing noise strength. At a critical noise strength a sharp transition in the behavior of the swarm occurs (see also Fig. 4).

The sequence of snapshots in Fig. 4 displays temporal evolution of the swarm when the noise intensity exceeds the breakdown threshold. Initially, the swarm is traveling and its shape is similar to that characteristic for the weaker noise (cf. Fig. 2). However, in the course of time the swarm slowly acquires a ring shape, with particles rotating around its center. This rotating ring structure corresponds to a state where translational motion of the entire swarm is already absent.

A different visualization of the process accompanying the breakdown of translational motion and the transition to a rotating swarm is chosen in Fig. 5. Here we show the trajectory of motion of the center of mass of the swarm (solid line) together with the trajectory of motion of one of its particles (dashed line). We see that, in a traveling swarm, the particles perform irregular oscillations around its instantaneous mass center. When the translational motion is terminated and the rotating ring is formed, such oscillations become transformed into rotations around the ring center.

To provide statistical description of particle motions in the traveling and rotating states of the swarm, angular momentum distributions $P(L)$ have been computed. For the state with translational motion, the distribution has a single central

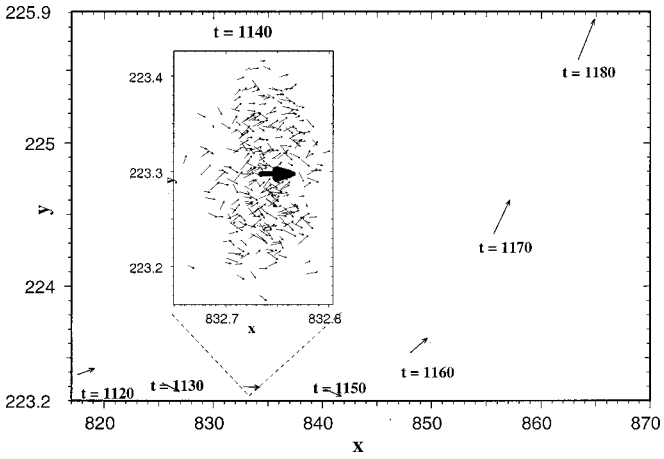

FIG. 2. Motion of the center of mass of the swarm (300 particles) within a certain time window for noise below a critical one. For $t=1140$ the corresponding snapshot of the swarm is shown. The big black arrow in the inset shows the mean swarm velocity; the noise intensity is $D=0.067$.

peak at $L=0$ [Fig. 6(a)]. In contrast to this, in the rotational state the distribution has two symmetrically placed peaks corresponding to a certain nonvanishing momentum [Fig. 6(b)]. Note that the particles inside the ring are rotating in both the clockwise and counterclockwise directions, and the numbers of particles rotating in each direction are approximately equal. Thus, the swarm does not rotate as a whole and its total angular momentum remains zero on the average. This behavior is a consequence of the fact that only long-range attractive interactions between particles are present in the considered model. It can be expected that, if short-range repulsive interactions are additionally introduced, the breakdown of the rotational symmetry in the ring would occur and one of the rotation directions would be selected [32].

IV. THE WEAK NOISE LIMIT

Our numerical simulations have shown that, for weak noise, the swarm is strongly squeezed in the direction along

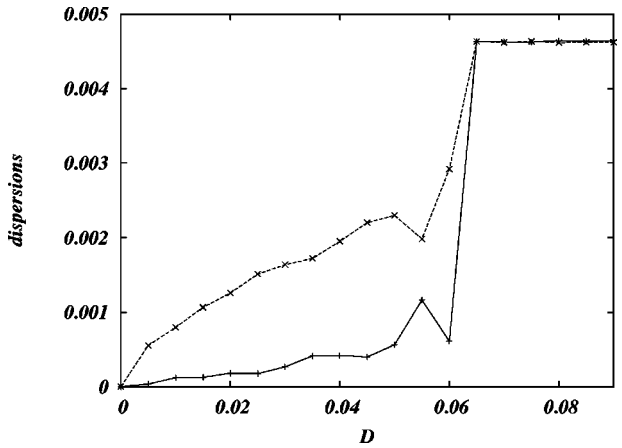

FIG. 3. Behavior of the longitudinal [solid line (+)] and transversal [dashed line (×)] dispersions of the swarm with increasing noise.

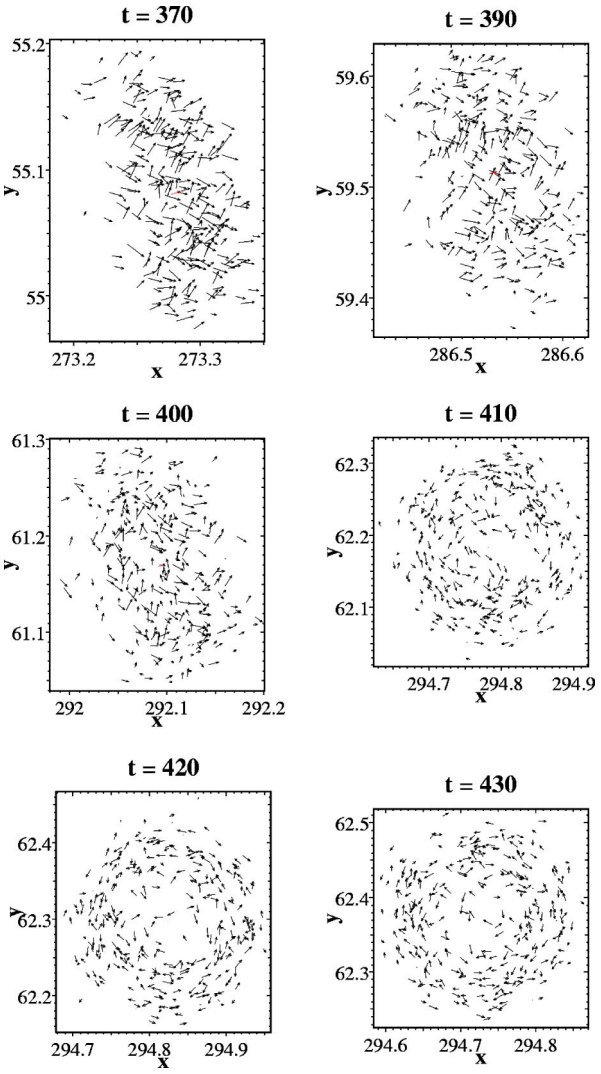

FIG. 4. Several sequential snapshots of a swarm with 300 particles during the transition from translational motion to the rotational mode; the noise intensity is $D=0.070$.

its center-of-mass motion and its longitudinal and transverse dispersions are strongly different. Below we derive approximate analytical expressions for $S_{\parallel}$ and $S_{\perp}$ in the limit of the small noise intensity $D \to 0$.

First, we note that in this limit the motion of the center of mass of the swarm will remain approximately linear within very long times or, in other words, the swarm velocity **V** remains approximately constant on the short time scales characteristic for the motions of individual particles inside the traveling swarm. We introduce the coordinate system in such a way that its x axis is parallel to the direction of the swarm motion and its y axis is orthogonal to it. The coordinates x_i and y_i of all particles forming the swarm can be written as $x_i = X + \delta x_i$ and $y_i = Y + \delta y_i$ where X and Y are the coordinates of the swarm center **R**. By our choice, we have $Y=0$, so that $y_i = \delta y_i$.

To derive the evolution equation for X, we notice that

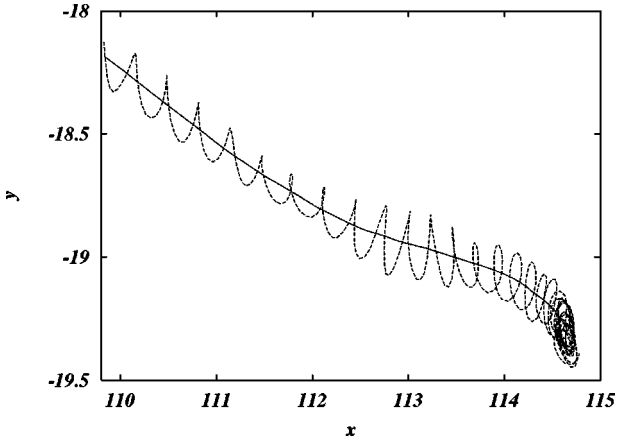

FIG. 5. The trajectories of the center of mass (solid line) and of a single particle (dashed line) are shown above the critical noise strength (for $D=0.070$).

$$X(t) = \frac{1}{N}\sum_{i=1}^{N} x_i(t) \equiv \langle x_i(t) \rangle . \tag{5}$$

Summing up the evolution equations for all x_i, we approximately obtain

$$\ddot{X} - (1 - \dot{X}^2)\dot{X} - 3\langle \dot{\delta x}_i^2 \rangle \dot{X} - \langle \dot{y}_i^2 \rangle \dot{X} = 0 \tag{6}$$

where we have neglected the terms with higher powers of the velocity fluctuations $\dot{\delta x}_i$ and $\dot{y}_i$.

In the statistical steady state, $\dot{X} = V = \text{const}$ and Eq. (6) is reduced to the equation

$$[(1 - V^2) - 3\langle \dot{\delta x}_i^2 \rangle - \langle \dot{y}_i^2 \rangle]V = 0 \tag{7}$$

determining the velocity of swarm motion in the presence of noise. Its solution for the traveling swarm ($V \neq 0$) is

$$V^2 = 1 - 3\langle \dot{\delta x}_i^2 \rangle - \langle \dot{y}_i^2 \rangle . \tag{8}$$

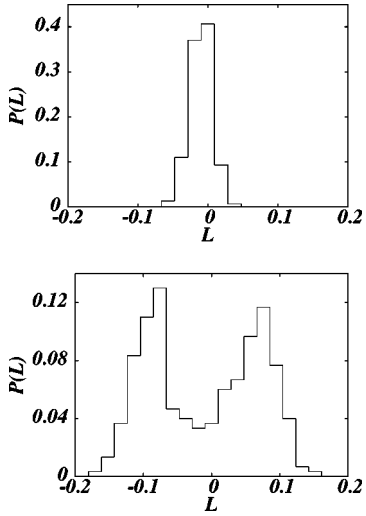

FIG. 6. Distribution of angular momenta of the particles (a) in the traveling ($D=0.067$) and (b) rotating ($D=0.070$) swarms.

The evolution equation for δx_i can be obtained by subtracting Eq. (6) from the equation for the variable x_i in the model (1). Keeping only the leading terms, linear in deviations from the mass center, we get

$$\ddot{\delta x}_i + 2\dot{\delta x}_i + a\delta x_i = \xi_i^x(t). \tag{9}$$

This is an evolution equation for a damped harmonic oscillator. Note that fluctuations of $x_i(t)$ are not coupled to the transverse component $y_i(t)$.

In a similar way, the evolution equation for the transverse deviations $y_i(t)$ can be obtained,

$$\ddot{y}_i - (1 - V^2)\dot{y}_i + 2V\dot{\delta x}_i\dot{y}_i + \dot{y}_i^3 + ay_i = \xi_i^y(t). \tag{10}$$

In this equation, we have retained nonlinear terms. This is done because such terms are essential for the damping of oscillations of the transverse component.

Indeed, if such terms were neglected, we would have

$$\ddot{y}_i - (1 - V^2)\dot{y}_i + ay_i = \xi_i^y(t). \tag{11}$$

Because, as follows from Eq. (8), we have $V^2 < 1$, oscillations in y_i will then exponentially grow with time. Thus, nonlinear terms play a principal role for transverse fluctuations and cannot be neglected even in the weak noise limit, in contrast to the corresponding terms for the longitudinal fluctuations δx_i.

As will be verified at the end of our derivation, the condition $\langle \dot{y}_i^2 \rangle \gg \langle \dot{\delta x}_i^2 \rangle$ holds in the weak noise limit. Therefore, the swarm velocity is mostly influenced by the transverse fluctuations and we have approximately $V^2 = 1 - \langle \dot{y}_i^2 \rangle$. Substituting this into Eq. (10), we get

$$\ddot{y}_i - (\langle \dot{y}_i^2 \rangle - \dot{y}_i^2)\dot{y}_i + ay_i = \xi_i^y(t). \tag{12}$$

The stochastic differential equation (12) does not include longitudinal fluctuations δx_i and, furthermore, fluctuations for different particles i are not coupled here. For subsequent analysis of this differential equation, we drop the indices and write it as

$$\ddot{y} - (\langle \dot{y}^2 \rangle - \dot{y}^2)\dot{y} + ay = \xi(t) \tag{13}$$

with $\langle \xi(t)\xi'(t) \rangle = 2D\delta(t - t')$. Here we have assumed that statistical averaging is equivalent to averaging over the ensemble.

The approximate solution for the probability distribution of variable y in the statistical stationary state can be derived for this equation assuming that the parameter a, determining the oscillation frequency, is large ($a \gg 1$). We introduce slowly varying amplitudes

$$y(t) = \eta(t)e^{i\omega t} + \eta^*(t)e^{-i\omega t} \tag{14}$$

where $\omega = \sqrt{a} \gg 1$. Substituting this into Eq. (13) and keeping only the resonant terms of the highest order in ω, we obtain a stochastic evolution equation for the transverse complex oscillation amplitudes

$$\dot{\eta} = \omega^2 \langle |\eta|^2 \rangle \eta - \frac{3}{2}\omega^2 |\eta|^2 \eta + \zeta(t) \tag{15}$$

where the complex-valued white noise $\zeta(t)$ has correlation functions

$$\langle \zeta(t) \rangle = 0, \quad \langle \zeta(t)\zeta(t') \rangle = 0, \quad \langle \zeta(t)\zeta^*(t') \rangle = \frac{D}{2\omega^2}\delta(t - t').$$

This stochastic Langevin equation corresponds to the following Fokker-Planck equation for the probability density $P = P(\eta, \eta^*, t)$:

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial \eta}\left[\omega\left(\langle |\eta|^2 \rangle \eta - \frac{3}{2}|\eta|^2 \right)\eta P \right] - \frac{\partial}{\partial \eta^*}\left[\omega\left(\langle |\eta|^2 \rangle \eta - \frac{3}{2}|\eta|^2 \right)\eta^* P \right] + \frac{1}{2\omega^2} D \frac{\partial^2 P}{\partial \eta\, \partial \eta^*}. \tag{16}$$

The stationary solution $\overline{P}$ of the Fokker-Planck equation reads

$$\overline{P} = \frac{1}{Z}\exp\left[-\frac{\omega^4}{D}(-4\langle |\eta|^2 \rangle |\eta|^2 + 3|\eta|^4) \right] \tag{17}$$

where the normalization constant Z is given by

$$Z = \int \exp\left[-\frac{\omega^4}{D}(-4\langle |\eta|^2 \rangle |\eta|^2 + 3|\eta|^4) \right] d^2\eta. \tag{18}$$

When the probability distribution is known, the second statistical moment can be calculated as

$$\langle |\eta|^2 \rangle = \int |\eta|^2 \overline{P}(\eta, \eta^*) d^2\eta. \tag{19}$$

Because the stationary probability distribution (17) depends on $\langle |\eta|^2 \rangle$, this is the equation which should be solved to determine this statistical moment. Let us substitute $\eta = \rho e^{i\phi}$ and $\omega^2 = a$. Then $|\eta| = \rho$ and $d^2\eta = \rho\, d\rho\, d\phi$. Therefore, Eq. (19) takes the form

$$\langle \rho^2 \rangle = \frac{\int_0^\infty \rho^3 \exp[-(a^2/D)(-4\langle \rho^2 \rangle \rho^2 + 3\rho^4)] d\rho}{\int_0^\infty \rho \exp[-(a^2/D)(-4\langle \rho^2 \rangle \rho^2 + 3\rho^4)] d\rho}. \tag{20}$$

Introducing the variable $u = \rho \langle |\eta|^2 \rangle^{-1/2}$, Eq. (20) is transformed to

$$1 = \frac{\int_0^\infty u^3 \exp[-\nu(-4u^2 + 3u^4)] du}{\int_0^\infty u \exp[-\nu(-4u^2 + 3u^4)] du}, \tag{21}$$

where

$$\nu = \frac{a^2}{D}\langle|\eta|^2\rangle^2. \tag{22}$$

Numerical solution of Eq. (21) yields $\nu \simeq 0.22$. When ν is known, Eq. (22) determines $\langle|\eta|^2\rangle$ as

$$\langle|\eta|^2\rangle = \nu^{1/2} D^{1/2} a^{-1}. \tag{23}$$

Using the definition (14) of variable η, we find that $S_\perp = \langle y^2\rangle = 2\langle|\eta|^2\rangle$. Thus, we finally obtain the analytical estimate for the transverse dispersion of the swarm in the weak noise limit,

$$S_\perp = \kappa \frac{D^{1/2}}{a}, \tag{24}$$

where the numerical coefficient is $\kappa = 2\nu^{1/2} \simeq 0.94$.

The longitudinal dispersion $S_\parallel = \langle \delta x^2\rangle$ is approximately determined by the linear stochastic differential equation (9). A straightforward derivation for $a \gg 1$ yields for this property the analytical estimate

$$S_\parallel = \frac{D}{2a}. \tag{25}$$

We see that in the limit $D \to 0$ the longitudinal dispersion is indeed much smaller than the transverse dispersion of the traveling swarm, as assumed in the above derivation.

Note that, for $a \gg 1$, statistical dispersions of transverse and longitudinal velocity fluctuations $W_\perp = \langle \dot{y}^2\rangle$ and $W_\parallel = \langle \delta \dot{x}^2\rangle$ are $W_\perp = aS_\perp$ and $W_\parallel = aS_\parallel$. Therefore, in the considered weak noise limit they are given by $W_\perp = \kappa D^{1/2}$ and $W_\parallel = D/2$. Comparing these analytical estimates with the simulations in Fig. 7, we find that they agree well with the corresponding numerical data. For the transverse dispersion, agreement is found in a wide interval of noise intensities. The analytical expression for the longitudinal dispersion holds, on the other hand, only for very weak noise.

V. CONCLUSIONS

We have studied statistical properties of localized swarms with long-range attractive interactions in two-dimensional media. Our numerical simulations show that the swarm is highly sensitive to the action of noise. Even very weak noise leads to strong dispersion of the swarm along the direction orthogonal to the direction of its translational motion. The approximate analytical theory predicts that the transverse dispersion of aswarm increases as $\sqrt{D}$ with the noise intensity D, whereas its longitudinal dispersion depends linearly on D in the limit $D \to 0$ and remains therefore much smaller in this limit. Hence, for weak noise the traveling swarm is strongly squeezed along the direction of its mass motion. This analytical result is confirmed by numerical simulations.

Increasing the noise intensity D, we find that translational motion breaks down when a certain critical intensity is reached. After the breakdown, the translational motion is stopped and instead the swarm goes into a rotational mode where the center of mass of the swarm shows only weak random motion. This behavior resembles the breakdown of translational motion which was previously seen for the one-dimensional system [40]. In contrast to the one-dimensional case, we could not, however, analytically treat this transition, because of the strong fluctuations in the transverse direction.

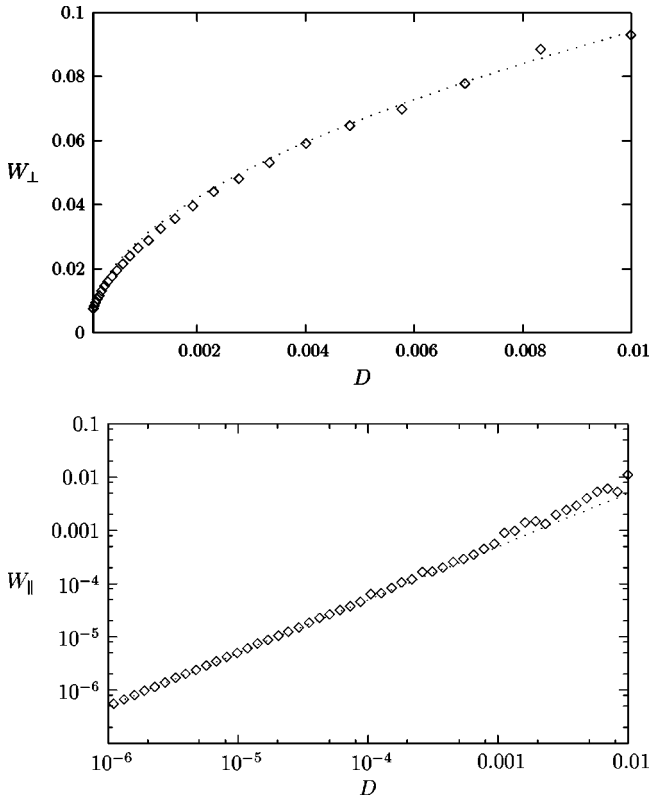

FIG. 7. Dispersions of swarm velocities in (a) transversal and (b) longitudinal directions as functions of the noise intensity. The symbols show the simulation data. The dotted lines are the theoretically predicted power law dependences.

Although our results are obtained in a model with harmonic attractive interactions, they are also applicable for models with finite-range attractive interactions between the particles, provided that the size of a localized swarm (i.e., the statistical dispersion of the coordinates of its particles with respect to the mass center) is much smaller than the interaction radius. In this situation, a harmonic approximation of the interaction potential can be applied. Moreover, similar effects can be expected for swarms in three-dimensional media. It would be interesting to see whether the discussed behavior is indeed characteristic for real biological swarms with long-range interactions between individual organisms, such as bird flocks or fish schools.

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